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Motivation:

Random ordinary differential equations (RODEs) are pathwise ODEs that contain a stochastic process in their vector field functions. They have been very much overshadowed by stochastic ordinary differential equations (SODEs), but there is, in fact, a close connection between RODEs and SODEs, which relates RODE and an SODE with the same solutions.

In this poster, 2 biological models with different type of noise processes are introduced. They are transformed into coupled RODE-SODE forms and simulated by averaged schemes, Itô-Taylor schemes and stochastic linear multi-step methods (SLMMs).

Lotka-Volterra model [3]:

$$\frac{dA}{dt} = (\epsilon_A - kB)A, \quad \frac{dB}{dt} = (-\epsilon_B + kA)B,$$

where A and B are the prey and predator compartments respectively.

In [3], the predatory rate k is considered to have a property of switching mechanism. Now introduce a

stochastic process

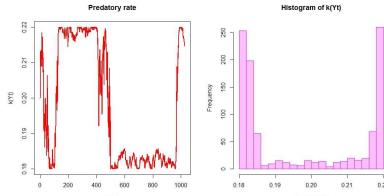
$$k(Y_t) = k_0 \left(1 - 2\nu \frac{Y_t}{1 + Y_t^2} \right),$$

where Y_t is a Wiener process, which describes switching effect.

Combining Y_t with 2-dimensional

prey-predator model gives 3-dimensional RODE-SODE system:

$$d \begin{pmatrix} A_t \\ B_t \\ Y_t \end{pmatrix} = \begin{pmatrix} (\epsilon_A - k(Y_t)B)A \\ (-\epsilon_B + k(Y_t)A)B \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dW_t.$$



HBV virus-kinetic model with spatial dependence [4]:

$$\frac{\partial T}{\partial t} = \lambda - aT - \beta TV, \quad \frac{\partial I}{\partial t} = \beta TV - \delta I, \quad \frac{\partial V}{\partial t} = d\Delta V + pI - cV,$$

where T , I and V are the compartments of target cells, infected cells and free viruses respectively and ΔV is the spatial derivative given by

$$\Delta V = \sum_{i=1}^2 \partial^2 V / \partial x_i^2.$$

The loss rate of infected cells δ is randomized by a stochastic process

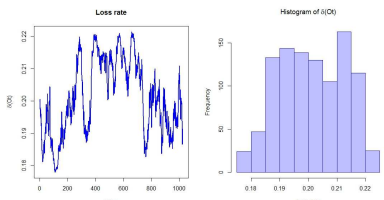
$$\delta(O_t) = \delta_0 \left(1 - \frac{2\nu}{\pi} \arctan(O_t) \right),$$

where O_t is an Ornstein-Uhlenbeck (OU) process.

Discretizing the above RPDEs with respect to space by method of lines and combining OU-process give a coupled RODE-SODE:

$$d \begin{pmatrix} T_{i,j} \\ I_{i,j} \\ V_{i,j} \\ O_t \end{pmatrix} = \begin{pmatrix} \lambda_{i,j} - aT_{i,j} - \beta T_{i,j}V_{i,j} \\ \beta T_{i,j}V_{i,j} - \delta(O_t)I_{i,j} \\ dD_{V_{i,j}} + pI_{i,j} - cV_{i,j} \\ \theta_1 - \theta_2 O_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \theta_3 \end{pmatrix} dW_t,$$

where $D_{V_{i,j}} = ((V_{i+1,j} - 2V_{i,j} + V_{i-1,j}) + (V_{i,j+1} - 2V_{i,j} + V_{i,j-1}))/2\Delta_x^2$ and e.g. $V_{i,j}$ is the value of V in i, j -th grid with equidistant partition size Δ_x .



Numerical Methods [1,2]:

Consider a RODE-SODE pair in the form:

$$d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} f(X_t, Y_t) \\ a(Y_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ b(Y_t) \end{pmatrix} dW_t$$

with an independent scalar Wiener process and $f \in \mathbb{R}^d$. Then the numerical methods can be derived in the same manner with SODE schemes.

1.0-, 1.5- and 2.0-order explicit Itô-Taylor schemes for k -th component of RODEs-part are given by

$$X_n^k = X_{n-1}^k + f^k(X_{n-1}, Y_{n-1}) \Delta_t + L^1 f^k(X_{n-1}, Y_{n-1}) I_{(1,0),t_{n-1}} + L^0 f^k(X_{n-1}, Y_{n-1}) I_{(0,0),t_{n-1}} + L^1 L^1 f^k(X_{n-1}, Y_{n-1}) I_{(1,1,0),t_{n-1}}.$$

Here, the differential operators L^0 and L^1 and the stochastic integrals $I_{(\cdot),t}$ are given by

$$L^0 U = \sum_{j=1}^d f^j(X, Y) \frac{\partial U}{\partial X_j} + a(Y) \frac{\partial U}{\partial Y} + \frac{1}{2} b(Y)^2 \frac{\partial^2 U}{\partial Y^2}, \quad L^1 U = b(Y) \frac{\partial U}{\partial Y},$$

$$I_{(0),t_{n-1}} = \int_{t_{n-1}}^{t_n} ds, \quad I_{(1),t_{n-1}} = \int_{t_{n-1}}^{t_n} dW_s, \quad I_{(1,0),t_{n-1}} = \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{s_1} dW_{s_2} ds_1.$$

Similarly, s -step 1.0-, 1.5- and 2.0-order SLMMs are

$$X_n^k = \sum_{l=1}^s \alpha_l X_{n-l}^k + \sum_{l=0}^s \beta_l f^k(X_{n-l}, Y_{n-l}) \Delta_t + \sum_{l=1}^s L^1 f^k(X_{n-l}, Y_{n-l}) (\gamma_l I_{(1,0),t_{n-1}} + \gamma_l^* I_{(1),t_{n-1}} \Delta_t) + \sum_{l=1}^s L^0 f^k(X_{n-l}, Y_{n-l}) (\delta_l I_{(0,0),t_{n-1}} + \delta_l^* I_{(0),t_{n-1}} \Delta_t) + \sum_{l=1}^s L^1 L^1 f^k(X_{n-l}, Y_{n-l}) (\epsilon_l I_{(1,1,0),t_{n-1}} + \epsilon_l^* I_{(1,1),t_{n-1}} \Delta_t).$$

The averaged schemes are obtained in the same manner. Typical example is the implicit averaged Euler scheme

$$X_n^k = X_{n-1}^k + f^k(X_n, \bar{Y}_n) \Delta_t,$$

where $\bar{Y}_n := \int_{(n-1)\Delta_t}^{n\Delta_t} Y_s ds / \Delta_t$ is the averaged noise within the vector field.

Results:

Figure 1 shows the results of 100 times simulation of Lotka-Volterra model.

- The averaged scheme shows 1.0-order convergence while the accuracy is relatively low comparing to higher order schemes.
- $L^1 L^1 f$ term in the 2.0-order schemes are very small and 1.5-order schemes show 2.0-order decay.
- No big difference in computational cost is observed between Itô-Taylor schemes and SLMMs.

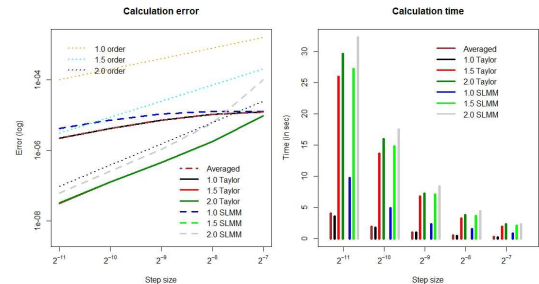


Figure 1: 100 times simulation results of Lotka-Volterra model

Figure 2 illustrates the 25 times simulation of HBV virus-kinetic model.

Each compartment was discretized into 20×20 grids, i.e. $i, j = 1, 2, \dots, 20$.

- $L^1 L^1 f$ term in the 2.0-order schemes are very small and 1.5-order schemes show 2.0-order decay.
- Now the system is of high dimension due to the spatial discretization and the difference in computational cost, especially between 1.5-order Itô-Taylor scheme and SLMM, is quite apparent.

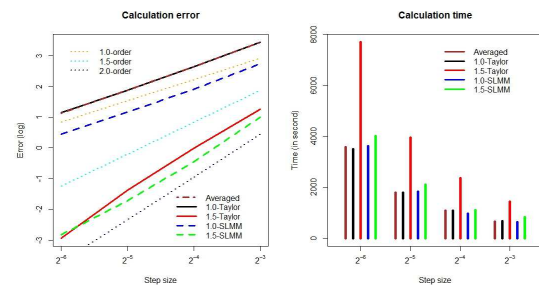


Figure 2: 25 times simulation results of HBV virus-kinetic model

Conclusion

- The SLMMs have big advantage in computational costs especially when they are applied to large systems.
- The implicit schemes are stable and can be applied to stiff systems or spatially discretized RODEs.
- Choosing appropriate type of noise is still an open question and further investigation will be necessary.